



The Open University

MS221/Specimen and Solutions

Course Examination
Exploring Mathematics

Time allowed: 3 hours

There are **TWO** parts to this paper.

In Part 1, you should attempt as many questions as you can.
In Part 2, credit will be given for answers to no more than TWO questions. If you submit attempts to more than two questions in Part 2, your best two scores will count towards your result.

Your answers to each part should be written in the answer books provided. You should start your answer to each question on a new page. You are advised not to cross through any work until you have replaced it with another solution to the same question.

Part 1 of the examination carries 72% of the available marks, and Part 2 carries 28%. In the examiners' opinion, most candidates would make best use of their time by finishing as much as they can of Part 1 before starting Part 2.

In most questions, some marks will be awarded for intermediate steps in the working. A correct answer not supported by working – for example, one taken from a calculator program – may not receive full credit.

At the end of the examination

Check that you have written your personal identifier and examination number on each answer book used. **Failure to do so will mean that your work cannot be identified.**

Put all your used answer books and your question paper together, with your signed desk record on top. Fix them all together with the fastener provided.

PART 1

Instructions

- (i) *You should attempt as many questions as you can in this part of the examination.*
- (ii) *Part 1 carries 72% of the available examination marks (6% of the marks to each question). The allocation of marks to parts of questions is shown on the right.*
- (iii) *You should record your answers to each question in the answer book(s) provided, beginning each question on a new page. You are strongly advised to show all your working, including any rough working.*

Question 1

Find a closed form for the sequence given by the following recurrence system:

$$u_0 = 3, \quad u_1 = -2, \quad u_{n+2} = 3u_{n+1} + 4u_n \quad (n = 0, 1, 2, \dots). \quad [6]$$

Question 2

This question concerns the curve with equation

$$3x^2 + 4y^2 = 24.$$

- (a) Show that the curve is a conic in standard position. [2]
- (b) Find the foci, directrices and eccentricity of this conic. [2]
- (c) Sketch the conic, showing the foci and directrices, and giving the coordinates of the points at which it meets the axes. [2]

Question 3

- (a) (i) Write down the rules for the isometries $t_{3,-2}$ and $q_{\pi/4}$. [1]
(ii) Determine the rule for the composite isometry $f = q_{\pi/4} \circ t_{3,-2}$. [2]
- (b) Let θ be the angle in the interval $(\frac{1}{2}\pi, \pi)$ for which $\sin \theta = \frac{1}{5}$. Determine the exact value of $\sin(2\theta)$. [3]

Question 4

- (a) Sketch, on the same axes, the graphs of $y = x$ and $y = f(x)$, where

$$f(x) = x^2 - x - 3.$$

The scale on each axis should run from -5 to 10 . [2]

- (b) Determine algebraically the fixed points of f . [2]

- (c) Use graphical iteration to determine the long-term behaviour of the iteration sequence given by

$$x_{n+1} = x_n^2 - x_n - 3 \quad (n = 0, 1, 2, \dots)$$

when $x_0 = 1$. This may be done on the graph drawn for part (a). [2]

Question 5

- (a) Find the matrices $\mathbf{Q}_{\pi/3}$ and $\mathbf{R}_{\pi/4}$ that represent the linear transformations $q_{\pi/3}$ and $r_{\pi/4}$, respectively, leaving your answers in surd form. [2]

- (b) Find the matrices of the linear transformations $q_{\pi/3} \circ r_{\pi/4}$ and $r_{\pi/4} \circ q_{\pi/3}$, being careful to identify which is which in your answer. [4]

Question 6

Let f be the linear transformation that maps $(1, 0)$ to $(1, 3)$ and $(0, 1)$ to $(-1, 2)$. Also, let g be the linear transformation that maps $(1, 0)$ to $(-2, 1)$ and $(0, 1)$ to $(1, 2)$.

- (a) Write down the matrices $\mathbf{A}$ and $\mathbf{B}$ that represent f and g , respectively. [2]

- (b) Find the matrix of the linear transformation that maps $(1, 3)$ to $(-2, 1)$ and $(-1, 2)$ to $(1, 2)$. [4]

Question 7

Differentiate each of the following functions. (There is no need to simplify your answers.)

- (a) $f(x) = \frac{\ln(x-1)}{\cos x} \quad (x \in (1, \frac{1}{2}\pi))$ [3]

- (b) $g(x) = \sqrt{\frac{x^3}{2} - \frac{2}{x^3}} \quad (x \in \mathbb{R}, x > 2)$ [3]

Question 8

- (a) Find the indefinite integral

$$\int \sec^2(4x) dx. \quad [3]$$

- (b) Find the indefinite integral

$$\int \frac{\sin x}{2 + \cos x} dx,$$

using the substitution $u = 2 + \cos x$. [3]

Question 9

Using standard Taylor series given in the Handbook, find the Taylor series about 0 for each of the following functions, as far as the fourth non-zero term.

(a) $f(x) = \frac{1}{1+x^3}$ [3]

(b) $g(x) = xe^{-3x}$ [3]

Question 10

- (a) Express the complex number below in the exponential form $re^{i\theta}$, where θ is the principal value of the argument:

$$2\sqrt{2} - 2\sqrt{2}i. \quad [3]$$

- (b) Hence find two complex numbers z that satisfy the equation

$$z^2 = 2\sqrt{2} - 2\sqrt{2}i,$$

giving your answers in the form $re^{i\theta}$. [3]

Question 11

- (a) Use Euclid's Algorithm to find the multiplicative inverse of 22 in $\mathbb{Z}_{57}$. [5]
- (b) Give an example of an integer in $\mathbb{Z}_{57}$, other than 0, which has no multiplicative inverse in $\mathbb{Z}_{57}$. [1]

Question 12

Consider the group $(G, *)$ whose incomplete Cayley table is given below.

$*$	p	q	r	s	t	u
p	q	s	t	p	u	r
q	s	p	u	q	r	t
r	u	t	s	r	q	p
s	p	q	r	s	t	u
t	r	u	$\square$	t	s	q
u	t	r	q	u	p	s

- (a) Which group element is equal to $t * r$? [1]
- (b) What is the identity element of G ? Give a brief reason for your answer. [3]
- (c) All groups of order 6 are isomorphic to either $(\mathbb{Z}_6, +_6)$ or $(S(\triangle), \circ)$. To which of these is $(G, *)$ isomorphic? Give a brief reason for your answer. [2]

PART 2

Instructions

- (i) *Credit will be given for answers to no more than **TWO** questions from this part of the examination.*
- (ii) *Each question in this part carries 14% of the total marks for the examination.*
- (iii) *You may answer the questions in any order. Write your answers in the answer book(s) provided, beginning each question on a new page.*
- (iv) *Show all your working.*

Question 13

In this question, L is the quadratic curve with equation

$$34x^2 - 24xy + 41y^2 = 50.$$

- (a) Show that L is $r_\theta(K)$, where K is the ellipse $\frac{1}{2}x^2 + y^2 = 1$ and $\theta \simeq 37^\circ$. [10]
- (b) Draw a sketch of the curve L , and indicate the equations of the lines that are its axes of symmetry. [4]

Question 14

- (a) Find the eigenvalues and eigenlines of the matrix $\mathbf{A} = \begin{pmatrix} 3 & 1 \\ 6 & 4 \end{pmatrix}$.
Write down one eigenvector for each eigenline. [4]
- (b) Write down a matrix $\mathbf{P}$ and a diagonal matrix $\mathbf{D}$ such that $\mathbf{A} = \mathbf{PDP}^{-1}$. [2]
- (c) Use your answer to part (b) to find the matrix $\mathbf{A}^4$. [5]
- (d) Describe in words the long-term behaviour of the iteration sequence generated by $\mathbf{A}$ with initial point
 - (i) $(1, -2)$; (ii) $(1, 1)$. [3]

Question 15

- (a) (i) By considering $\ln x$ as the product $(\ln x) \times 1$, and using integration by parts, show that

$$\int \ln x \, dx = x(\ln x - 1) + c,$$

where c is an arbitrary constant.

[4]

- (ii) Hence find the area under the graph of the function $f(x) = \ln x$ from $x = 1$ to $x = 3$, giving your answer to 3 decimal places.

[3]

- (b) (i) Using the result of part (a)(i), find the indefinite integral

$$\int (\ln x)^2 \, dx.$$

[4]

- (ii) Hence find the volume of revolution obtained when the region under the graph of the function $f(x) = \ln x$, from $x = 1$ to $x = 3$, is rotated about the x -axis. Give your answer to 3 decimal places.

[3]

Question 16

- (a) In this part of the question, m and n are integers.

Proposition (A) is as follows:

If $2m^2 + n^2$ is divisible by 9, then m and n are both divisible by 3. (A)

Proposition (B) is the converse of proposition (A).

One of propositions (A) and (B) is true, and the other is false.

- (i) Write down a statement of proposition (B). [1]

- (ii) State which of propositions (A) and (B) is false, and give a counter-example to demonstrate this. [3]

- (iii) Give a proof that the remaining proposition is true. [4]

- (b) Prove, using mathematical induction, that

$$\frac{1}{1 \times 2} + \frac{1}{2 \times 3} + \frac{1}{3 \times 4} + \cdots + \frac{1}{n(n+1)} = \frac{n}{n+1}$$

for all positive integers n .

[6]

[END OF QUESTION PAPER]

Solutions

PART 1

Comment You should attempt all questions in this part.

Question 1

We solve the auxiliary equation

$$r^2 - 3r - 4 = 0 \quad [1 \text{ for equation}]$$

or $(r - 4)(r + 1) = 0,$

so $r = 4$ or $r = -1$. [1 for roots]

The general solution is

$$u_n = A4^n + B(-1)^n, \quad n = 0, 1, 2, \dots \quad [1 \text{ for method}]$$

To find A and B we use the first two terms:

$$u_0 = 3 = A + B,$$

$$u_1 = -2 = 4A - B.$$

So $5A = 1$, hence $A = \frac{1}{5}$, and $B = \frac{14}{5}$. [2 for constants]

The required closed form is

$$u_n = \frac{1}{5}4^n + \frac{14}{5}(-1)^n, \quad n = 0, 1, 2, \dots \quad [1 \text{ for closed form}]$$

Comment

See Handbook page 45

In the auxiliary equation, $p = 3$ and $q = 4$ (see Handbook). In the last line of the solution, notice the need to contain -1 in brackets, to avoid any confusion when powers of it are taken. (Odd powers will give -1 , even powers will give $+1$.)

Question 2

(a) Dividing throughout by 24 gives

$$\frac{x^2}{8} + \frac{y^2}{6} = 1. \quad [1 \text{ for method}]$$

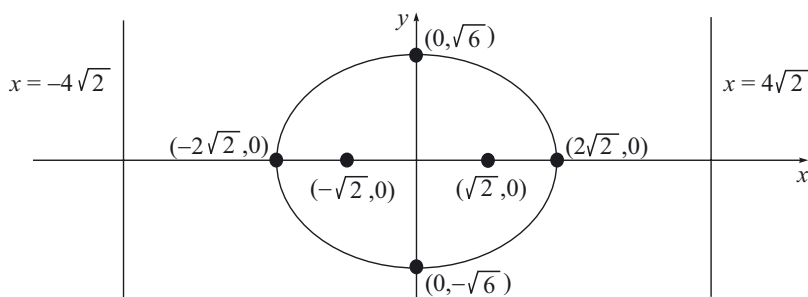
This is the equation of an ellipse in standard position with $a = \sqrt{8} = 2\sqrt{2}$
and $b = \sqrt{6}$. [1 for constants]

(b) The eccentricity is $e = \sqrt{1 - b^2/a^2} = \sqrt{1 - 6/8} = \frac{1}{2}$. [1 for eccentricity]

So the foci are at $(\pm ae, 0) = (\pm\sqrt{2}, 0)$, [$\frac{1}{2}$ for foci]

and the directrices are $x = \pm a/e$, that is, $x = \pm 4\sqrt{2}$. [$\frac{1}{2}$ for directrices]

- (c) Since, to three significant figures, $\sqrt{2} = 1.41$, $2\sqrt{2} = 2.83$, $4\sqrt{2} = 5.66$ and $\sqrt{6} = 2.45$, the diagram is as follows. [2 for sketch]



Comment

See Handbook page 46

Always divide through by a number that will give the value 1 on the right-hand side. Compare the result with the table of standard conics in the Handbook, which also gives the coordinates of the foci and the equations of the directrices for conics in standard position.

Question 3

- (a) (i) $t_{3,-2} : (x, y) \mapsto (x + 3, y - 2)$, [1/2]
 $q_{\pi/4} : (x, y) \mapsto (y, x)$. [1/2]
 (ii) From part (i), $q_{\pi/4} \circ t_{3,-2} : (x, y) \mapsto (y - 2, x + 3)$. [2]
- (b) Using $\cos^2 \theta + \sin^2 \theta = 1$ from the Handbook,

$$\cos^2 \theta = 1 - \left(\frac{1}{5}\right)^2 = \frac{24}{25} \quad \left[\frac{1}{2}\right]$$

so

$$\cos \theta = \pm \sqrt{\frac{24}{25}} = \pm \frac{2}{5} \sqrt{6}. \quad \left[\frac{1}{2}\right]$$

Since θ is in the interval $(\frac{1}{2}\pi, \pi)$, $\cos \theta$ is negative, so

$$\cos \theta = -\frac{2}{5} \sqrt{6}. \quad \left[\frac{1}{2}\right]$$

Then

$$\begin{aligned} \sin(2\theta) &= 2 \sin \theta \cos \theta \quad (\text{from the Handbook}) & \left[\frac{1}{2}\right] \\ &= 2 \times \frac{1}{5} \times \left(-\frac{2}{5} \sqrt{6}\right) \\ &= -\frac{4}{25} \sqrt{6}. & [1] \end{aligned}$$

Comment

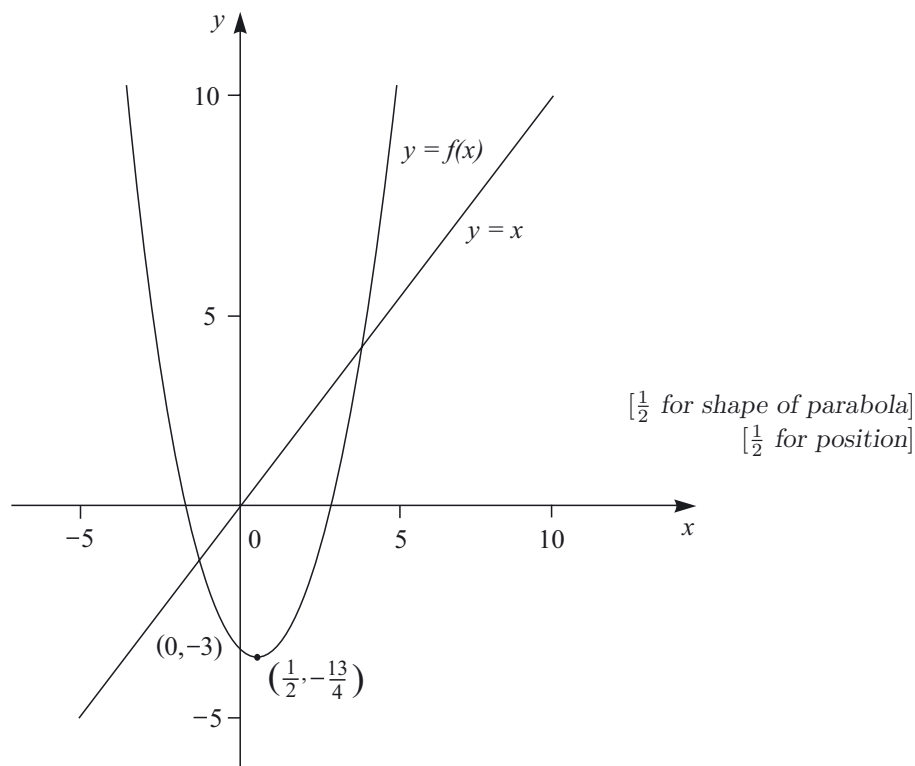
See Handbook page 48

- (a) (i) See the Handbook for general rules. Do not forget to substitute numerical values for the trigonometric ratios in the rule for q_θ .
 (ii) Remember that with a composite isometry, it is the isometry on the right that is carried out first, not that on the left.
- (b) Always check which quadrant contains the angle. This will affect the signs of the trigonometric ratios.

Question 4

- (a) First we write $f(x)$ in completed-square form:

$$\begin{aligned} f(x) &= x^2 - x - 3 \\ &= \left(x - \frac{1}{2}\right)^2 - \frac{1}{4} - 3 = \left(x - \frac{1}{2}\right)^2 - \frac{13}{4}. \end{aligned} \quad [1 \text{ for completed-square form}]$$



- (b) The fixed point equation is

$$x^2 - x - 3 = x,$$

that is,

$$x^2 - 2x - 3 = 0$$

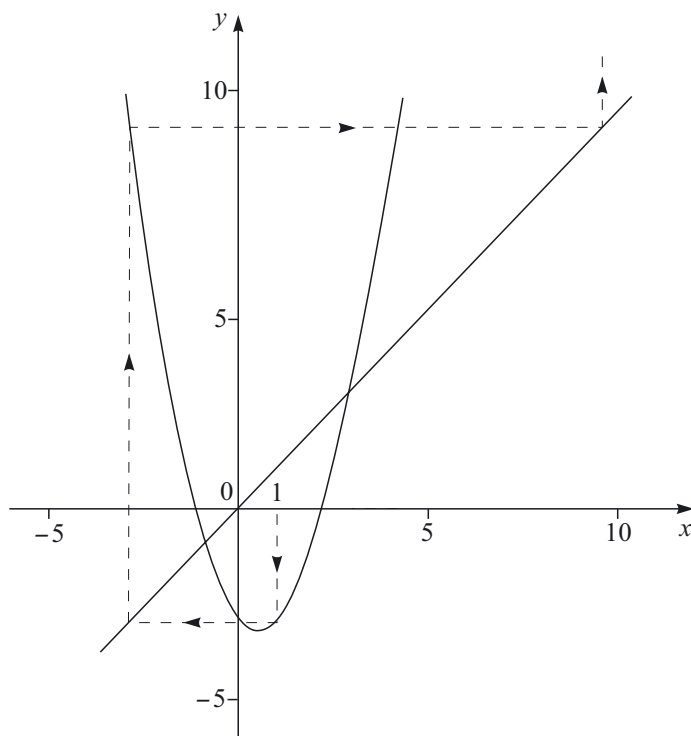
or $(x - 3)(x + 1) = 0.$

So the fixed points are 3 and -1 .

[1 for equation]

[1 for roots]

(c)



[1 for trace]

So, in the long term, the sequence tends to infinity.

[1 for result]

Comment

See Handbook pages 41, 56, 37

- (a) It is worth completing the square to find the vertex of the parabola before trying to sketch it. The examiners are looking for a sketch (NOT plotting) of a parabola, with a correctly positioned axis of symmetry and with its vertex at the correct point.
- (b) If you cannot see the factors of the fixed point equation immediately, it is acceptable to use the formula for solving a quadratic equation.
- (c) Remember that you do NOT need to re-draw the sketch.

Question 5

$$(a) \mathbf{Q}_{\pi/3} = \begin{pmatrix} \cos\left(\frac{2\pi}{3}\right) & \sin\left(\frac{2\pi}{3}\right) \\ \sin\left(\frac{2\pi}{3}\right) & -\cos\left(\frac{2\pi}{3}\right) \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}, \quad [1]$$

$$\mathbf{R}_{\pi/4} = \begin{pmatrix} \cos\left(\frac{\pi}{4}\right) & -\sin\left(\frac{\pi}{4}\right) \\ \sin\left(\frac{\pi}{4}\right) & \cos\left(\frac{\pi}{4}\right) \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}. \quad [1]$$

- (b) $q_{\pi/3} \circ r_{\pi/4}$ has matrix

$$\begin{aligned} \mathbf{Q}_{\pi/3} \mathbf{R}_{\pi/4} &= \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \\ &= \begin{pmatrix} \frac{\sqrt{3}-1}{2\sqrt{2}} & \frac{1+\sqrt{3}}{2\sqrt{2}} \\ \frac{1+\sqrt{3}}{2\sqrt{2}} & \frac{1-\sqrt{3}}{2\sqrt{2}} \end{pmatrix}. \end{aligned} \quad [2]$$

$r_{\pi/4} \circ q_{\pi/3}$ has matrix

$$\begin{aligned}\mathbf{R}_{\pi/4} \mathbf{Q}_{\pi/3} &= \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix} \\ &= \begin{pmatrix} \frac{-1-\sqrt{3}}{2\sqrt{2}} & \frac{\sqrt{3}-1}{2\sqrt{2}} \\ \frac{\sqrt{3}-1}{2\sqrt{2}} & \frac{\sqrt{3}+1}{2\sqrt{2}} \end{pmatrix}. \end{aligned} \quad [2]$$

Comment

See Handbook pages 59, 60

- (a) Remember that surd form means leaving square root signs in the answers; do not convert to decimals.
- (b) It is important to remember to write the matrices down in the order in which the linear transformations are written. In general, matrix multiplication is not commutative (i.e. the result can be different if the matrices are written down in reverse order).

Question 6

(a) $\mathbf{A} = \begin{pmatrix} 1 & -1 \\ 3 & 2 \end{pmatrix},$ [1]

$\mathbf{B} = \begin{pmatrix} -2 & 1 \\ 1 & 2 \end{pmatrix}.$ [1]

- (b) The required linear transformation is $g \circ f^{-1}$, which has matrix $\mathbf{BA}^{-1}$. [1 for method]

$\mathbf{A}^{-1} = \frac{1}{5} \begin{pmatrix} 2 & 1 \\ -3 & 1 \end{pmatrix},$ [2 for inverse matrix]

so

$$\begin{aligned}\mathbf{BA}^{-1} &= \frac{1}{5} \begin{pmatrix} -2 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ -3 & 1 \end{pmatrix} \\ &= \frac{1}{5} \begin{pmatrix} -7 & -1 \\ -4 & 3 \end{pmatrix}. \end{aligned} \quad [1 \text{ for product}]$$

Comment

See Handbook page 60

- (a) Since the images of the unit vectors $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ are known, the matrices can be written down directly.
- (b) Notice that the coordinates given are the images under the linear transformations f and g in part (a). It is first necessary to apply f^{-1} to transform $(1, 3)$ and $(-1, 2)$ back to $(1, 0)$ and $(0, 1)$, respectively, so that g can be applied. Be sure to divide by the determinant of the matrix $\mathbf{A}$ when finding $\mathbf{A}^{-1}$.

Question 7

(a) $f(x) = \frac{\ln(x-1)}{\cos x}$ gives [1 for method]

$$f'(x) = \frac{(x-1)^{-1} \cos x + \ln(x-1) \sin x}{\cos^2 x},$$
 [2 for differentiations]

using the Quotient Rule and the Composite Rule (for $\ln(x-1)$).

(b) $g(x) = \left(\frac{x^3}{2} - \frac{2}{x^3}\right)^{1/2}$ gives [1 for method]

$$g'(x) = \frac{1}{2} \left(\frac{x^3}{2} - \frac{2}{x^3}\right)^{-1/2} \left(\frac{3x^2}{2} + \frac{6}{x^4}\right),$$
 [1 for $\frac{1}{2}(\)^{-1/2}$]

using the Composite Rule.

[1 for derivative of bracket]

Comment

See Handbook page 65

In terms of Leibniz notation:

(a) apply the Quotient Rule with $v = \cos x$ and $u = \ln(x - 1)$;

(b) apply the Composite Rule, with $u = x^3/2 - 2/x^3$ and $y = u^{1/2}$.

Question 8

(a) The integral is [1 for method]

$$\int \sec^2(4x) dx = \frac{1}{4} \tan(4x) + c,$$
 [2 for integral]

using either inspection (from the table of integrals in the Handbook) or integration by substitution (with $u = 4x$).

(b) Taking $u = 2 + \cos x$, we have $du/dx = -\sin x$, and so

$$\begin{aligned} \int \frac{\sin x}{2 + \cos x} dx &= - \int \frac{1}{u} du && [1 \text{ for substitution}] \\ &= -\ln u + c && [1 \text{ for integration}] \\ &= c - \ln(2 + \cos x). && [1 \text{ for back-substitution}] \end{aligned}$$

Comment

See Handbook pages 68, 70

(a) Use the table of indefinite integrals. Remember to add an arbitrary constant.

(b) Do not forget to 'substitute du for $\frac{du}{dx} dx$ ', express the final answer in terms of x , and include a constant c .

Question 9

(a) From the Handbook,

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \dots$$

Replacing x by $-x^3$ gives

[1 for method]

$$\begin{aligned} \frac{1}{1+x^3} &= 1 - x^3 + (-x^3)^2 + (-x^3)^3 + \dots && [1 \text{ for correct handling of } -\text{ve signs}] \\ &= 1 - x^3 + x^6 - x^9 + \dots && [1 \text{ for answer}] \end{aligned}$$

(b) Using the series for e^x from the Handbook, and putting $-3x$ in place of x , gives

[1 for method]

$$\begin{aligned} xe^{-3x} &= x \left(1 - 3x + \frac{1}{2}(-3x)^2 + \frac{1}{6}(-3x)^3 + \dots\right) \\ &= x - 3x^2 + \frac{9}{2}x^3 - \frac{9}{2}x^4 + \dots && [2 \text{ for algebra}] \end{aligned}$$

Comment

See Handbook page 74

(a) This is obtained from the Taylor series expansion of $\frac{1}{1-x}$. Replace x by $-x^3$, which should be enclosed in brackets to avoid errors in sign.

(b) First write down the Taylor series expansion for e^x . Then substitute $-3x$, in brackets, for x . Notice that the powers apply to both the -3 and x , so that the terms will alternate in sign. Finally, multiply each term by x .

Question 10

- (a) The modulus is given by $r^2 = (2\sqrt{2})^2 + (-2\sqrt{2})^2 = 16$, so $r = 4$. [1 for modulus]
The principal value of the argument is the solution of the equations

$$\cos \theta = \frac{2\sqrt{2}}{4} = \frac{1}{\sqrt{2}}, \quad \sin \theta = -\frac{2\sqrt{2}}{4} = -\frac{1}{\sqrt{2}}, \quad [1 \text{ for } \cos \theta, \sin \theta]$$

in the interval $(-\pi, \pi]$. Hence $\theta = -\pi/4$, and the exponential form of $2\sqrt{2} - 2\sqrt{2}i$ is $4e^{-i\pi/4}$. [1 for answer]

- (b) Since $z^2 = 4e^{-i\pi/4}$, the modulus of z is $\sqrt{4} = 2$. [1 for modulus]

For the argument θ of z , we have $2\theta = -\pi/4 + 2m\pi$, where $m = 0, 1$; that is, $\theta = -\pi/8$ or $\theta = 7\pi/8$. [1 for values of θ]

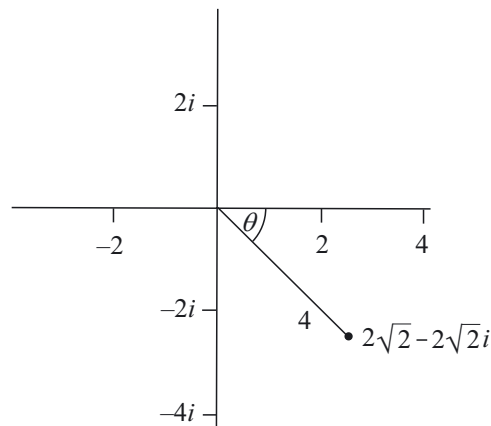
So the two solutions for z are

$$z = 2e^{-i\pi/8} \quad \text{and} \quad z = 2e^{7i\pi/8}. \quad [1 \text{ for answers}]$$

Comment

See Handbook pages 79, 80

- (a) $re^{i\theta} = r \cos \theta + ir \sin \theta$, so $r \cos \theta = 2\sqrt{2}$ and $r \sin \theta = -2\sqrt{2}$. Therefore r^2 is the sum of the squares of these two terms (since $\cos^2 \beta + \sin^2 \beta = 1$). This gives $r^2 = 8 + 8 = 16$, and so $r = 4$.



$$\cos \theta = \frac{2\sqrt{2}}{4} = \frac{1}{\sqrt{2}}, \quad \sin \theta = -\frac{2\sqrt{2}}{4} = -\frac{1}{\sqrt{2}}, \quad \text{so } \theta = -\frac{\pi}{4}.$$

- (b) See the Handbook: Finding roots.

Question 11

- (a) $57 = 2 \times 22 + 13$
 $22 = 1 \times 13 + 9$
 $13 = 1 \times 9 + 4$
 $9 = 2 \times 4 + 1$ [2]

Working backwards,

$$\begin{aligned} 1 &= 9 - 2 \times 4 \\ &= 9 - 2 \times (13 - 9) \\ &= -2 \times 13 + 3 \times 9 \\ &= -2 \times 13 + 3 \times (22 - 13) \\ &= 3 \times 22 - 5 \times 13 \\ &= 3 \times 22 - 5 \times (57 - 2 \times 22) \\ &= 13 \times 22 - 5 \times 57. \end{aligned} \quad [2]$$

So $13 \times 22 = 5 \times 57 + 1$, and 13 is the required inverse. [1 for answer]

- (b) 3 or 19 (or a multiple of 3 or 19 less than 57). [1]

Comment*See Handbook pages 83, 82*

- (a) *See the Handbook for Euclid's Algorithm. Remember that when working backwards, each expression on the RHS should equal 1. It is worth checking occasionally.*
- (b) *Notice that $57 = 3 \times 19$. No number in $\mathbb{Z}_{57}$ which is a multiple of 3 or 19 has a multiplicative inverse in $\mathbb{Z}_{57}$.*

Question 12

- (a) $t * r = p$ [1]
- (b) The identity element is s (its row and column repeat the order of the table borders). [1 for s]
[2 for row and column]
- (c) $(G, *)$ is isomorphic to $(S(\Delta), \circ)$ since both have four self-inverse elements, whereas $\mathbb{Z}_6$ has only two self-inverse elements. [1 for group]
[1 for reason]

Comment*See Handbook pages 85, 86*

- (a) *Remember that each row and each column of a group table should contain each element of the group exactly once. In the row labelled t , the element p is missing, which is also true for the column headed r . So $t * r = p$.*
- (b) *Each element is unchanged when it is combined with the identity element.*
- (c) *See the Cayley tables in the Handbook. Compare the orders of the groups (numbers of elements). Where these are the same, compare the numbers of self-inverse elements.*

PART 2**Comment** *Attempt 2 out of the 4 questions in this part.***Question 13**

- (a) For this quadratic curve, $A = 34$, $B = -24$ and $C = 41$. [1 for method]

A suitable rotation r_θ is obtained from

$$\tan(2\theta) = \frac{B}{A - C} = \frac{-24}{34 - 41} = \frac{24}{7}. \quad [1 \text{ for } \tan(2\theta)]$$

Thus $\theta = \frac{1}{2} \arctan(24/7)$, which is approximately 37° . Then [1 for θ]

$$\cos(2\theta) = \frac{1}{\sqrt{1 + \tan^2(2\theta)}} = \frac{1}{\sqrt{1 + (24/7)^2}} = \frac{7}{25} \quad [1]$$

and

$$\sin(2\theta) = \frac{\tan(2\theta)}{\sqrt{1 + \tan^2(2\theta)}} = \frac{24/7}{\sqrt{1 + (24/7)^2}} = \frac{24}{25}. \quad [1]$$

So

$$\cos^2 \theta = \frac{1 + \cos(2\theta)}{2} = \frac{16}{25}, \quad \sin^2 \theta = \frac{1 - \cos(2\theta)}{2} = \frac{9}{25} \quad [1]$$

and

$$\sin \theta \cos \theta = \frac{1}{2} \sin(2\theta) = \frac{12}{25}.$$

Hence

$$\begin{aligned} A' &= A \cos^2 \theta + B \sin \theta \cos \theta + C \sin^2 \theta \\ &= 34 \times \frac{16}{25} - 24 \times \frac{12}{25} + 41 \times \frac{9}{25} \\ &= \frac{625}{25} \\ &= 25 \end{aligned} \quad [1]$$

and

$$\begin{aligned}
 C' &= A \sin^2 \theta - B \sin \theta \cos \theta + C \cos^2 \theta \\
 &= 34 \times \frac{9}{25} + 24 \times \frac{12}{25} + 41 \times \frac{16}{25} \\
 &= \frac{1250}{25} \\
 &= 50.
 \end{aligned}
 \tag{1}$$

Thus the equation of K is

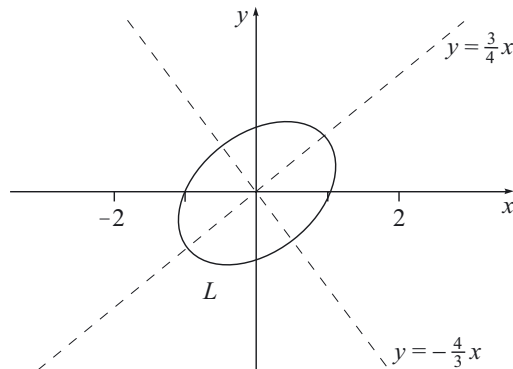
$$25x^2 + 50y^2 = 50, \tag{1 for substitution}$$

that is,

$$\frac{1}{2}x^2 + y^2 = 1, \tag{1 for standard form}$$

as required.

- (b) For the ellipse K , we have $a = \sqrt{2}$ and $b = 1$. The axes of symmetry of L are $y = (\tan \theta)x$ and $y = -\frac{1}{\tan \theta}x$, i.e. $y = \frac{3}{4}x$ and $y = -\frac{4}{3}x$. [1 for new axes]
[1 for $\tan \theta$]

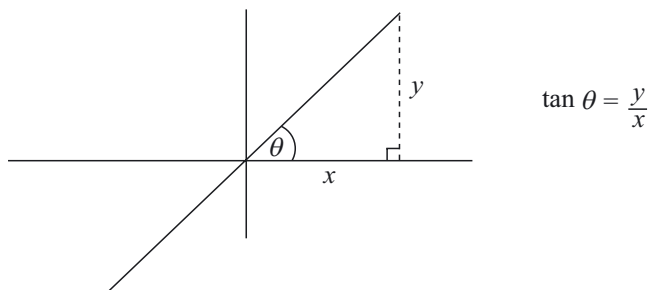


[2 for sketch and labelling features]

Comment

See Handbook page 49

- (a) When finding θ remember to have your calculator in degree mode. The method for finding the coefficients is given in the Handbook. These should be found by using trigonometric formulas, as in the solution, and not by calculating the trigonometric ratios directly from the angle 37° . This would give only approximate values, because 37° is only an approximation to θ .
- (b) The line through the origin making an angle θ with the positive direction of the x -axis has gradient $\tan \theta$, and so has equation $y = (\tan \theta)x$.



The axes of symmetry are obtained by rotating the x -axis and y -axis (the axes of symmetry of the ellipse in standard position) through the angle $\theta \simeq 37^\circ$. Since $\cos^2 \theta = \frac{16}{25} = \left(\frac{4}{5}\right)^2$, $\sin^2 \theta = \frac{9}{25} = \left(\frac{3}{5}\right)^2$ and θ lies in the first quadrant, we have $\tan \theta = \frac{3}{4}$.

Question 14

- (a) The matrix
- $\mathbf{A}$
- has characteristic equation

$$k^2 - 7k + 6 = 0. \quad [1 \text{ for equation}]$$

This factorises as $(k - 6)(k - 1) = 0$, so the eigenvalues of the matrix are 6 and 1. [1 for values]

For $k = 6$, the eigenvector equations are

$$\begin{aligned} -3x + y &= 0, \\ 6x - 2y &= 0. \end{aligned}$$

That is, the eigenline is $y = 3x$ and one eigenvector is $\begin{pmatrix} 1 \\ 3 \end{pmatrix}$. [1]

For $k = 1$, the eigenvector equations are

$$\begin{aligned} 2x + y &= 0, \\ 6x + 3y &= 0. \end{aligned}$$

That is, the eigenline is $y = -2x$ and one eigenvector is $\begin{pmatrix} 1 \\ -2 \end{pmatrix}$. [1]

(b) $\mathbf{P} = \begin{pmatrix} 1 & 1 \\ 3 & -2 \end{pmatrix}$ with $\mathbf{D} = \begin{pmatrix} 6 & 0 \\ 0 & 1 \end{pmatrix}$. [2]

(c) $\mathbf{A} = \mathbf{PDP}^{-1}$, so $\mathbf{A}^4 = (\mathbf{PDP}^{-1})^4 = \mathbf{PD}^4\mathbf{P}^{-1}$: [1 for method]

$$\mathbf{P}^{-1} = -\frac{1}{5} \begin{pmatrix} -2 & -1 \\ -3 & 1 \end{pmatrix} = \frac{1}{5} \begin{pmatrix} 2 & 1 \\ 3 & -1 \end{pmatrix}, \quad [1 \text{ for inverse matrix}]$$

$$\mathbf{D}^4 = \begin{pmatrix} 6^4 & 0 \\ 0 & 1^4 \end{pmatrix} = \begin{pmatrix} 1296 & 0 \\ 0 & 1 \end{pmatrix}, \quad [1]$$

so

$$\begin{aligned} \mathbf{A}^4 &= \begin{pmatrix} 1 & 1 \\ 3 & -2 \end{pmatrix} \begin{pmatrix} 1296 & 0 \\ 0 & 1 \end{pmatrix} \frac{1}{5} \begin{pmatrix} 2 & 1 \\ 3 & -1 \end{pmatrix} \\ &= \frac{1}{5} \begin{pmatrix} 1 & 1 \\ 3 & -2 \end{pmatrix} \begin{pmatrix} 2592 & 1296 \\ 3 & -1 \end{pmatrix} \quad [1 \text{ for first product}] \\ &= \frac{1}{5} \begin{pmatrix} 2595 & 1295 \\ 7770 & 3890 \end{pmatrix} \\ &= \begin{pmatrix} 519 & 259 \\ 1554 & 778 \end{pmatrix} \quad [1 \text{ for answer}] \end{aligned}$$

- (d) (i) The point $(1, -2)$ lies on the eigenline with eigenvalue 1, so it remains fixed under the iterative process, and the sequence is constant. [1]
- (ii) The point $(1, 1)$ does not lie on either eigenline. Since both eigenvalues are greater than zero, all points of the sequence lie on the same side of each eigenline as $(1, 1)$ does. Since $\max\{1, 6\} = 6 > 1$, the sequence moves away from the origin. Since $6 > 1$, the sequence (x_n, y_n) moves so that $y_n/x_n \rightarrow 3$ (the gradient of the eigenline for eigenvalue 6) as $n \rightarrow \infty$. [2]

Comment

See Handbook pages 62, 63

- (a) *The characteristic equation is given in the Handbook. Remember that for each eigenvalue there is no unique eigenvector, but the arithmetic is usually easier if one of the components is taken as 1. It is also sometimes helpful to avoid fractions.*
- (b) *Remember that the order of the columns in $\mathbf{P}$ must match the order of the eigenvalues in $\mathbf{D}$.*
- (c) *Notice the first line of the solution – this is important. The question specifically says Use your answer to part (b), so you will receive few, if any, marks if you multiply four copies of $\mathbf{A}$ together.*
- (d) *If the initial point lies on an eigenline, then the points in the iteration sequence will remain on that eigenline, but with the distance from the origin multiplied by the magnitude of the eigenvalue for each iteration. See the Handbook.*

Question 15

- (a) (i) Let $f(x) = \ln x$ and $g'(x) = 1$,
so $f'(x) = 1/x$ and $g(x) = x$. [2 for four components]

Then, using the formula $\int f g' = fg - \int f' g$, we have

$$\begin{aligned}\int \ln x \, dx &= (\ln x)x - \int \left(\frac{1}{x}\right) x \, dx \\ &= x \ln x - \int 1 \, dx \\ &= x \ln x - x + c \quad [2 \text{ for integration}] \\ &= x(\ln x - 1) + c.\end{aligned}$$

- (ii) Since $f(x) \geq 0$ for all $x \in [1, 3]$, the area is

$$\begin{aligned}\int_1^3 \ln x \, dx &= [x(\ln x - 1)]_1^3 \quad [1 \text{ for method}] \\ &= 3(\ln 3 - 1) - 1(0 - 1) \\ &= 3 \ln 3 - 2 \\ &= 1.296 \quad (\text{to 3 d.p.}). \quad [2 \text{ for calculation}]\end{aligned}$$

- (b) (i) Use integration by parts once more. [1 for method]

Let $f(x) = (\ln x)^2$ and $g'(x) = 1$,
so $f'(x) = 2(\ln x)(1/x)$ and $g(x) = x$. [1 for four components]

Then we have

$$\begin{aligned}\int (\ln x)^2 \, dx &= (\ln x)^2 x - \int 2 \ln x \left(\frac{1}{x}\right) x \, dx \\ &= x(\ln x)^2 - 2 \int \ln x \, dx \quad [1 \text{ for this expression}] \\ &= x(\ln x)^2 - 2x(\ln x - 1) + c \quad (\text{from part (a)(i)}) \\ &= x((\ln x)^2 - 2 \ln x + 2) + c. \quad [1 \text{ for integration}]\end{aligned}$$

- (ii) The required volume is

$$\begin{aligned}\pi \int_1^3 (\ln x)^2 \, dx &= \pi [x((\ln x)^2 - 2 \ln x + 2)]_1^3 \quad [1 \text{ for method}] \\ &= \pi [3((\ln 3)^2 - 2 \ln 3 + 2) - 2] \\ &= 3.233 \quad (\text{to 3 d.p.}). \quad [2 \text{ for calculations}]\end{aligned}$$

Comment

See Handbook page 70

- (a) Remember that if you have trouble obtaining a given answer, it does not stop you from using the result given elsewhere in the question.
(b) (i) An alternative approach would be to set $u = \ln x$ and $dv/dx = \ln x$.
(ii) Do not forget the π !

Question 16

- (a) (i) The converse of proposition (A) is:

If m and n are both divisible by 3, then $2m^2 + n^2$ is divisible by 9. (B) [1]

- (ii) Proposition (A) is false. [1]

Counter-example: If $m = 2$ and $n = 1$, then neither m nor n is divisible by 3, but $2m^2 + n^2 = 9$ is divisible by 9. [2]

(iii) Proof that proposition (B) is true:

If m and n are both divisible by 3, then $m = 3k$ and $n = 3l$,
for some integers k and l . [1 for assumption]

Then we have

$$2m^2 + n^2 = 18k^2 + 9l^2 = 9(2k^2 + l^2),$$
 [2 for algebra]

which is divisible by 9 since $2k^2 + l^2$ is an integer. [1 for conclusion]

(b) Let $p(n)$ be the variable proposition

$$\frac{1}{1 \times 2} + \frac{1}{2 \times 3} + \cdots + \frac{1}{n(n+1)} = \frac{n}{n+1}.$$

When $n = 1$, the left-hand side is $\frac{1}{1(1+1)} = \frac{1}{2}$ and the right-hand side

is $\frac{1}{1+1} = \frac{1}{2}$. Since these are equal, $p(1)$ is true. [1 for case $n = 1$]

Assume that $p(k)$ is true, that is,

$$\frac{1}{1 \times 2} + \frac{1}{2 \times 3} + \frac{1}{3 \times 4} + \cdots + \frac{1}{k(k+1)} = \frac{k}{k+1}.$$
 [1 for assumption]

Then we have

$$\begin{aligned} \frac{1}{1 \times 2} + \frac{1}{2 \times 3} + \frac{1}{3 \times 4} + \cdots + \frac{1}{k(k+1)} + \frac{1}{(k+1)(k+2)} \\ = \frac{k}{k+1} + \frac{1}{(k+1)(k+2)} \\ = \frac{k(k+2) + 1}{(k+1)(k+2)} \\ = \frac{(k+1)^2}{(k+1)(k+2)} \\ = \frac{k+1}{k+2} \\ = \frac{k+1}{(k+1)+1}. \end{aligned}$$
 [1 for method] [2 for algebra]

Hence the implication $p(k) \Rightarrow p(k+1)$ is true for all k in $\mathbb{N}$.

We can then deduce, by mathematical induction, that $p(n)$ is true
for all n in $\mathbb{N}$. Hence the result holds for all positive integers n . [1 for conclusion]

Comment

See Handbook page 88

(a) (i) To find the converse, mentally interchange if and then in the statement.

(iii) Note that a number is divisible by 9, say, if and only if it can be written
as $9k$ where k is an integer.

(b) After you have written down the assumption $p(k)$, it is worth writing down
 $p(k+1)$, the statement that you have to prove. You get this by replacing k
with $(k+1)$ in $p(k)$. Here, it is

$$\frac{1}{1 \times 2} + \frac{1}{2 \times 3} + \cdots + \frac{1}{(k+1)(k+2)} = \frac{k+1}{(k+1)+1} = \frac{k+1}{k+2}.$$

Here, to simplify the algebra in an expression like $\frac{k}{k+1} + \frac{1}{(k+1)(k+2)}$, do
not be tempted to multiply out all the brackets. Always look for common
factors. In this case, the denominator of each term contains $k+1$ so, when
the terms are combined, one term in the denominator will need to be $k+1$.
See the Handbook for a model answer.